

# More studies on Metamaterials Mimicking de Sitter space

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**Abstract:** We estimate the dominating frequencies contributing to the Casimir energy in a cavity of meta- materials mimicking de Sitter space, by solving the eigenvalue problem of Maxwell equations. It turns out the dominating frequencies are the inverse of the size of the cavity, and the degeneracy of these frequencies also explains our previous result on the unusually large Casimir energy. Our result suggests that carrying out the experiment in laboratory is possible theoretically.

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## 1. Introduction

The Casimir energy is one of the important predictions in quantum field theory and continues to be source of inspiration for theoretical as well as experimental work [1]. It is the regularized difference between two energies: one is the zero point energy of the electromagnetic field in a finite cavity and the other is that in an infinite background. Generically, the expression of the Casimir energy depends on the details of the cavity including properties of its bulk and its boundary. The earliest work [2] on the Casimir energy and the following up studies [3, 4, 5, 6, 7, 8, 9, 10, 11] all reported a result that the energy density of the Casimir energy is inversely proportional to the forth power of the typical size of the cavity. Applying this result to the universe, this kind of the Casimir energy can not be taken as a possible origin of dark energy, since  $1/L^4$  is too small compared with the observed dark energy density [12, 13] if  $L$  is chosen to be the a typical size of universe.

With doubt about the applicability of the previous results to de Sitter space, we carried out a calculation on the Casimir energy of electromagnetic field in the static patch of de Sitter space carefully [14]. We obtained a drastically different Casimir energy which is proportional to the size of the event horizon taking the same form as holographic dark energy [15, 16]. Then basing on the recent theoretical and experimental development of metamaterials [17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27], we proposed to design metamaterials mimicking de Sitter (along the line of studying cosmological phenomenon in laboratory, some other researches have been carried out including the negative phase velocity of electromagnetic wave in de Sitter, mimicking black hole and cosmic string by metamaterials [28, 29, 30]). Different from the usual ones, the permittivity and permeability parameters both have divergent compo-

nents tangent to the boundary. This unusual fact leads to a brand new Casimir energy in the cavity coinciding with the the result from the gravity side with the Planck scale is replaced by some microscopic scale in metamaterials. We encourage experimentalists to make such metamaterials and measure the predicted Casimir energy. This work will have significant implication to cosmology.

Apparently, it seems difficult to carry out such an experiment, because the Casimir energy is expressed as a sum of all the frequencies, meaning that each frequency contributes a part to this energy, while metamaterials have frequency dispersion, in other words, the designed permittivity and permeability is effective only to frequencies in certain band. However, this difficulty can be circumvented as we will uncover a fact that there is a typical frequency whose contribution to Casimir energy is dominating. Thus a cavity of metamaterials effective at this typical frequency is sufficient to mimic de Sitter space and induces the Casimir energy predicted by theoretical calculation.

We shall show that the typical dominating frequency is  $\omega \sim 1/L$ , where  $L$  is the size of the cavity, and the dominating angular quantum number is  $l \sim L/d$ ,  $d$  is the short distance cut-off. These two numbers conspire to give the Casimir energy  $L/d^2$ , the same order of our previous result [14].

The rest of this paper is organized as follows. In sect.2, we uncover the fact that the Casimir energy has a typical contributing frequency. We estimate the typical frequency in metamaterials mimicking de Sitter in sect.3. We conclude in sect.4.

## 2. General discussion on the typical frequency of Casimir energy

Physically, it is conceivable that there is a typical contributing frequency to the Casimir energy. According to its definition, the Casimir energy measures the difference between vacuum energy in a finite cavity and that in infinite background. The former is usually contributed by a discrete spectrum and the latter comes from a continuous one. Thus the Casimir measures the difference between discrete and continuous ones. When frequency is large, the discrete spectrum approaches a continuous one, and their contributions cancel with each other; for small frequencies, their contribution is also negligible. Since the contribution from very large and very small frequencies is tiny, there should be some intermediate scale at which the difference between discrete spectrum and continuous spectrum is maximum. Then the frequency at this scale is the typical frequency.

As a heuristic example, we read off this typical frequency from the process of computing Casimir energy in the static Einstein's universe. In this case, the Casimir energy is given by [31]

$$E_c = \frac{1}{2a_0} \sum_{n=0}^{\infty} n^3 - \frac{a_0^3}{2} \int_0^{\infty} \omega^3 d\omega, \quad (1)$$

where the discrete spectrum consists of  $\omega = n/a_0$  with degeneracy  $n^2$ , and  $a_0$  is the radius of Einstein universe. Reparameterizing  $\omega$  by  $t/a_0$ , then using Abel-Plana formula

$$\sum_{n=0}^{\infty} F(n) - \int_0^{\infty} dt F(t) = \frac{1}{2} F(0) + i \int_0^{\infty} dt \frac{F(it) - F(-it)}{e^{2\pi t} - 1}, \quad (2)$$

eq.(1) is equal to

$$E_C = \frac{1}{a_0} \int_0^{\infty} dt \frac{t^3}{e^{2\pi t} - 1}. \quad (3)$$

This result is very interesting because it appears that Casimir energy is contributed by frequencies satisfying a distribution similar to blackbody. We read off from this formula that typical

frequency is at  $t \approx 1/2\pi$  corresponding to  $\omega \approx 1/(2\pi a_0)$ . However, in the general three dimensional cases of physical interest, the discussion should take into account complications due to at least two reasons. The first one is that it is hard to find out the exact expression of the discrete frequencies of electromagnetic wave in a finite cavity which are eigenvalues of a complicated partial differential equation deduced from Maxwell equations. The second is that in three dimensions, one usually has two integers  $(n, l)$  to denote the discrete frequencies where  $n$  is radial quantum number and  $l$  is angular quantum number, thus Abel-Plana formula should be utilized repeatedly. Although it is difficult to derive an exact result, estimations are always possible under some reasonable assumptions. In the next section, we will try to solve the eigenvalue problem of Maxwell equations in the metamaterials designed to mimic de Sitter and estimate the typical frequency of the Casimir energy. Our estimation is based on an assumption that the radial quantum number  $n$  is frozen then the Casimir energy comes mostly from frequencies with smallest  $n$ . This is because in a finite cavity a frequency usually grows linearly with  $n$ , thus its contribution is suppressed exponentially by the blackbody factor. As a check of this assumption, the large  $n$  behavior of the eigenvalue will be given. It is indeed a linear function of  $n$ .

### 3. typical frequency of Casimir energy in metamaterial mimicking de Sitter

The metamaterials mimicking de Sitter space is designed with the following permittivity and permeability [14]

$$\epsilon^{\tilde{r}\tilde{r}} = \mu^{\tilde{r}\tilde{r}} = L^2 \frac{\sin^2(\tilde{r}/L)}{\cos(\tilde{r}/L)} \sin \theta, \quad \epsilon^{\theta\theta} = \mu^{\theta\theta} = \frac{\sin \theta}{\cos(\tilde{r}/L)}, \quad \epsilon^{\varphi\varphi} = \mu^{\varphi\varphi} = \frac{1}{\cos(\tilde{r}/L) \sin \theta}. \quad (4)$$

where  $(\tilde{r}, \theta, \varphi)$  denote the spherical coordinates. In terms of the Cartesian coordinates

$$\epsilon^{ij} = \mu^{ij} = \frac{1}{\cos(\tilde{r}/L)} \left( \delta^{ij} - \left( \frac{L^2}{\tilde{r}^2} \sin^2(\tilde{r}/L) - 1 \right) \frac{x^i x^j}{\tilde{r}^2} \right). \quad (5)$$

The event horizon at  $\tilde{r} = \pi L/2$  now becomes the boundary of a cavity of metamaterials.

The Maxwell equations in inhomogeneous medium are

$$\nabla_i E^i = 0, \quad \nabla_i H^i = 0, \quad (6)$$

$$\partial_t E^i - \frac{\epsilon^{ijk}}{\sqrt{\gamma}} \partial_j H_k = 0, \quad \partial_t H^i + \frac{\epsilon^{ijk}}{\sqrt{\gamma}} \partial_j E_k = 0, \quad (7)$$

where  $\gamma^{ij}$  is the optical metric related to permittivity and permeability through  $\epsilon^{ij} = \mu^{ij} = \sqrt{\gamma} \gamma^{ij}$ ,  $\gamma = \det(\gamma_{ij})$  and  $\nabla_i$  denotes the covariant derivative with respect to  $\gamma^{ij}$ , and all the indices are raised and lowered by  $\gamma^{ij}$ . To keep the realness of the frequency of electromagnetic wave and the finiteness of energy, the following boundary conditions are imposed

$$E_\theta|_{r=L-d} = E_\varphi|_{r=L-d} = 0. \quad (8)$$

These boundary conditions are acceptable physically, since the photons emitted from the center of de Sitter space will travel an infinite amount of time to arrive at the horizon or they can never reach there as seen by any static observer.

To solve Maxwell equations we adopt Newman-Penrose formalism [32]. That is to use four null vectors reexpress the Maxwell tensor  $F_{\mu\nu}$  as

$$F_{\mu\nu} = 2[\phi_1(n_{[\mu}l_{\nu]} + m_{[\mu}m_{\nu]}^*) + \phi_2 l_{[\mu}m_{\nu]} + \phi_0 m_{[\mu}^*n_{\nu]}] + c.c., \quad (9)$$

where “[ ]” denotes the antisymmetrization, and “c.c” means the complex conjugate. The convention about  $\phi_s$  and the null vectors is given by

$$\phi_0 = F_{\mu\nu} l^\mu m^\nu, \quad \phi_1 = \frac{1}{2} F_{\mu\nu} (l^\mu n^\nu + m^{*\mu} m^\nu), \quad \phi_2 = F_{\mu\nu} m^{*\mu} n^\nu, \quad (10)$$

with

$$\begin{aligned} l^\mu &= \left( \frac{1}{1-r^2/L^2}, 1, 0, 0 \right), \quad m^\mu = \frac{1}{\sqrt{2}r} \left( 0, 0, 1, \frac{i}{\sin \theta} \right) \\ n^\mu &= \left( \frac{1}{2}, -\frac{1-r^2/L^2}{2}, 0, 0 \right), \quad m^{*\mu} = \frac{1}{\sqrt{2}r} \left( 0, 0, 1, \frac{-i}{\sin \theta} \right). \end{aligned} \quad (11)$$

To solve Maxwell equations conveniently, we have adopted a coordinate system different from that appearing in (4), but we will transform back to the old coordinates when finding the typical frequency. Then after some standard steps [33], we obtain

$$\phi_1 = e^{-i\omega t} Y_l^m(\theta, \varphi) R(r), \quad (12)$$

where  $Y_l^m(\theta, \varphi)$  is the spherical harmonic function satisfies the following equation

$$\left[ \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta) + \frac{\partial_\varphi^2}{\sin^2 \theta} + l(l+1) \right] Y_l^m(\theta, \varphi) = 0, \quad (13)$$

and  $R(r)$  satisfies

$$\left[ r^2 \left( 1 - \frac{r^2}{L^2} \right) \partial_r^2 + 4r \left( 1 - \frac{3r^2}{2L^2} \right) \partial_r + \frac{r^2 \omega^2}{\left( 1 - \frac{r^2}{L^2} \right)} - \frac{6r^2}{L^2} + 2 - l(l+1) \right] R(r) = 0. \quad (14)$$

This equation possesses two independent solutions and the one of physical interest is

$$R(r) = r^{-2} \left( 1 + \frac{L}{r} \right)^{-l-1} \left( 1 + \frac{L}{r} \right)^{-i\omega L/2} \left( 1 - \frac{L}{r} \right)^{i\omega L/2} F(l+1, l+1+i\omega L, 2l+2, \frac{2r}{L+r}), \quad (15)$$

it is regular at  $r=0$  and goes back to flat space result  $j_l(\omega r)/r$  when  $L \rightarrow \infty$ . The solution of  $\phi_0$  and  $\phi_2$  can also be found, but to find out  $\omega$  solving  $\phi_1$  is enough.

Recall that in an empty spherical cavity the two independent electromagnetic modes are TE and TM corresponding to  $E_r = 0$  and  $H_r = 0$  respectively. Here the situation is similar. In the TE modes,  $E_r = 0$  the electrowave is transverse. From the definition of  $\phi_s$  eq.(10), we read  $\phi_1 = -\frac{1}{2}(E_r + iH_r) = -\frac{i}{2}H_r$ . Combining the following Maxwell equations and boundary conditions (8)

$$i\omega H_r = \frac{1}{r^2 \sin \theta} (\partial_\theta E_\varphi - \partial_\varphi E_\theta), \quad E_\theta|_{r=L-d} = E_\varphi|_{r=L-d} = 0, \quad (16)$$

We deduce that

$$\phi_1|_{r=L-d} = 0. \quad (17)$$

Since  $d \ll L$ ,  $L-d \sim L$ , the boundary conditions can be imposed on the  $r \rightarrow L$  behavior of  $\phi_1$ . When  $r \rightarrow L$ , the radial part of  $\phi_1$  has the following asymptotic form

$$R(r) \sim \frac{\Gamma(-i\omega L)}{\Gamma(l+1-i\omega L)} \left( \frac{1-r/L}{1+r/L} \right)^{i\omega L/2} + c.c. \quad (18)$$

Then eq.(17) requires that

$$\text{Re}\left(\frac{\Gamma(-i\omega L)}{\Gamma(l+1-i\omega L)}\left(\frac{1-r/L}{1+r/L}\right)^{i\omega L/2}\right)|_{r=L-d}=0, \quad (19)$$

In the coordinate system eq.(5), the cut-off  $d$  is imposed on the physical radial coordinate  $\tilde{r}$  (4), the cut-off on  $r$  is therefore  $d^2/(2L)$ , the above condition amounts to

$$\text{Re}\left(\frac{\Gamma(-i\omega L)}{\Gamma(l+1-i\omega L)}\left(\frac{1-r/L}{1+r/L}\right)^{i\omega L/2}\right)|_{r=L-d^2/2L}=0 \quad (20)$$

which determines value of  $\omega$ . For small  $l$ , we can solve eq.(20) to pick out the lowest  $\omega$  when  $\ln(4L^2/d^2) \gg 1$  (This is guaranteed by the fact that in usual metamaterials  $d$  is nanometer, and  $L$  is 1cm) and the other corresponds to very large  $n$ . The results are exhibited below.

1)  $l = 0$ . We find that (20) leads to

$$\sin\left[\frac{\omega L}{2}\ln(4L^2/d^2)\right] = 0 \Rightarrow \omega = \frac{2n\pi}{L\ln(4L^2/d^2)}, n = 1, 2, \dots \quad (21)$$

2)  $l = 1$ . Then (20) implies that

$$\omega L = \tan\left(\frac{\omega L}{2}\ln(4L^2/d^2)\right). \quad (22)$$

Since  $\ln(4L^2/d^2) \gg 1$ , the lowest  $\omega \approx 2\pi/L\ln(4L^2/d^2)$ .

Based on the above results, we infer that the Casimir energy from the frequencies corresponding to small  $l$  is of the order  $1/L\ln(4L^2/d^2)$  which is much smaller than  $L/d^2$  predicted in [14]. So the typical frequency cannot be around  $1/L\ln(4L^2/d^2)$ . To estimate the typical frequency, we observe that for  $l \gg 1$  the Stirling formula can be used to reexpress eq.(20) as

$$\text{Re}[\Gamma(-i\omega L)(dl/2L)^{i\omega L}] = 0. \quad (23)$$

We see that a critical  $l$  denoted by  $l_c$  emerges at  $l_c = 2L/d$  (Indeed  $l_c \gg 1$ , so our estimation is reasonable). For  $l \gg l_c$ , the term  $(dl/2L)^{i\omega L}$  becomes highly oscillating, and the  $\omega$  satisfying (23) approaches a continuous distribution whose effect is canceled by that from the infinite background. For  $l \ll l_c$ , this case is just what we discussed before, their contribution is subleading. Therefore the dominating contribution to the Casimir energy can only come from  $l \sim l_c$ . When  $l \sim l_c$ , the corresponding frequency is around  $1/L$  since in this case the constant appearing (23) is of order 1. Go one step further, we estimate the contribution to the Casimir energy from frequencies corresponding to  $l \sim l_c$ . For  $l_c/2 < l < 3l_c/2$ , for each  $l$  the degeneracy is  $2l+1$ ,

$$E_c \sim \sum_{l_c/2}^{3l_c/2} (2l+1)1/L \sim l_c^2/L = L/d^2. \quad (24)$$

This result is the same order as we obtained in [14] by a different method. It is a strong support to our estimation about the typical frequency.

In the TM modes, the magnetic wave is transverse

$$H_r = 0, \quad \phi_l = -\frac{1}{2}E_r. \quad (25)$$

Combined with the Gauss law and the boundary conditions (8)

$$[\partial_r(r^2 \sin \theta E_r) + \frac{\sin \theta}{1-\frac{r^2}{L^2}}\partial_\theta E_\theta + \frac{1}{(1-\frac{r^2}{L^2})\sin \theta}\partial_\phi E_\phi]|_{r=L-d^2/2L}=0 \quad (26)$$

We deduce that

$$\partial_r(r^2\phi_1)|_{r=L-d^2/2L} = \partial_r(r^2E_r)|_{r=L-d^2/2L} = 0. \quad (27)$$

After some simplification, this condition is transformed to

$$\text{Im}\left(\frac{\Gamma(-i\omega L)}{\Gamma(l+1-i\omega L)}\left(\frac{1-r/L}{1+r/L}\right)^{i\omega L/2}\right)|_{r=L-d^2/2L} = 0. \quad (28)$$

Then the process of finding frequency  $\omega$  satisfying above condition is the same as before and we list the results as follows

1)  $l = 0$  the frequency is given by

$$\cos\left[\frac{\omega L}{2} \ln(4L^2/d^2)\right] = 0 \Rightarrow \omega = \frac{(2n+1)\pi}{L \ln(4L^2/d^2)}, n = 0, 1, \dots \quad (29)$$

2)  $l = 1$ .

$$\omega L = -\cot\left(\frac{\omega L}{2} \ln(4L^2/d^2)\right). \quad (30)$$

For  $\ln(2L/d) \gg 1$ , the lowest frequency is

$$\omega \approx \pi/L \ln(4L^2/d^2). \quad (31)$$

3) The typical frequency can be read from

$$\text{Im}[\Gamma(-i\omega L)(dL/2L)^{i\omega L}] = 0. \quad (32)$$

Since the structure of eq.(32) and eq.(23) is similar, the typical frequency is still of the order  $1/L$  with the critical  $l_c$  given by  $2L/d$ .

We emphasize that to estimate the typical frequency we have assumed that the contribution from  $\omega$  with large radial quantum number  $n$  is suppressed exponentially. As a check of this assumption, we present the expression of  $\omega$  in large  $n$  limit. Both TE and TM modes have the following frequencies

$$\omega \approx \frac{n\pi}{L \ln(4L^2/d^2)}. \quad (33)$$

Thus for  $n$  is large,  $\omega$  grow with  $n$  linearly and their effects will be suppressed by the black body factor.

To end this section, we propose that the measurable quantity for such experiments would be the Casimir force. From eq.(24), we get

$$F_C \sim -\frac{1}{d^2}. \quad (34)$$

It is remarkable that this attractive force is relatively large compare with the usual Casimir force. Thus, it is more easily to measure the new Casimir force for a cavity with small cutoff. We also notice that the unusually large Casimir force is related to the permittivity and the permeability at  $\tilde{r} = \frac{\pi}{2}(L-d)$  (4)

$$\varepsilon = \mu \sim \frac{L}{d}. \quad (35)$$

where  $d$  is some microcosmic cutoff to keep the permittivity and the permeability finite but relatively large.

#### **4. Conclusion**

We estimate the typical frequency of the vacuum fluctuations in metamaterials mimicking de Sitter and find it proportional to  $1/L$ , the size of the cavity. Assuming  $d$  be 1 nanometer, and  $L$  be 1cm, then the typical wavelength is about 1cm.

With our estimation of the typical frequency and the typical angular quantum number, we also have an intuitive understanding of our Casimir energy formula.

We hope that one day the metamaterials suggested by us can be made with appropriate size and effective for the corresponding typical frequency, then the predicted brand new Casimir force can be measured, this experiment is important for study cosmology in laboratory.

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